Lecture 8
Motion in central force fields II.

8.1 Orbit in a central force field

In Lecture 7 we have seen the basic procedure for solving the equation of motion of a body in a central force field. We can obtain $r(t)$ and $\vartheta(t)$ as the function of time by integrating (see Eq. (7.28)):

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}}$$

Let us now derive a differential equation for the shape of the orbit, $r(\vartheta)$, ignoring how the body moves along the orbit in time. Using the conservation of angular momentum, Eq. (7.26a), we can relate the time derivative to the derivative with respect to ϑ:

$$mr^2\dot{\vartheta} = \ell \quad \Rightarrow \quad \ell\, dt = mr^2\, d\vartheta \quad \Leftrightarrow \quad \frac{d}{dt} = \frac{\ell}{mr^2}\frac{d}{d\vartheta} \tag{8.1}$$

Rewriting the equation of motion for r, Eq. (7.21), in terms of ϑ-derivatives we get

$$\frac{\ell}{r^2}\frac{d}{d\vartheta}\left(\frac{\ell}{mr^2}\frac{dr}{d\vartheta}\right) - \frac{\ell^2}{mr^3} = F(r) \tag{8.2}$$

Introducing the variable

$$u \equiv \frac{1}{r}, \qquad dr = -\frac{du}{u^2}$$

equation (8.2) will take the simpler quasilinear form

$$\frac{d^2u}{d\vartheta^2} + u = -\frac{m}{\ell^2}\frac{d}{du}V\left(\frac{1}{u}\right) \tag{8.3}$$

Eq. (7.28) can also be written in terms of ϑ and u:

$$\vartheta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}} + \vartheta_0 \tag{8.4}$$

$$\vartheta = -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - u^2}} + \vartheta_0 \tag{8.5}$$

Power law potentials of the form $V(r) = ar^{n+1} \Rightarrow F(r) \propto r^n$ have particular importance. The gravitational potential and Hooke's law are both power laws. Substituting a $V(r) = ar^{n+1}$ into Eq. (8.5) we get

$$\vartheta = -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2ma}{\ell^2}u^{-n-1} - u^2}} + \vartheta_0 \tag{8.6}$$

This integral can be expressed in terms of trigonometric functions for $n = 1, -2, -3$. We will discuss the solution for $n = -2$ in section 8.2.

8.1.1 Stability of a circular orbit

Let us derive the conditions for a *stable* circular orbit. Recall that $m\ddot{r} + \frac{d}{dr}V'(r) = 0$, where $V'(r) = \frac{\ell^2}{2mr^2} + V(r)$ is the effective potential (7.31). Thus the condition for a circular orbit with radius r_0 is

$$\left.\frac{dV'}{dr}\right|_{r=r_0} = 0 \quad \Leftrightarrow \quad \frac{\ell^2}{mr_0^3} = \left.\frac{dV}{dr}\right|_{r=r_0} = -F(r_0) \tag{8.7}$$

For a given angular momentum ℓ, the effective potential $V'(r)$ reaches its minimum or maximum at the radius of the circular orbit r_0 (see figures 8.1 and 8.2).

What happens if a body moving on a circular orbit is slightly perturbed? The orbit will be **stable** if $V'(r)$ is convex in r_0, i.e. it has a minimum (Fig. 8.1); and **unstable** if $V'(r)$ is concave in r_0, i.e. it has a

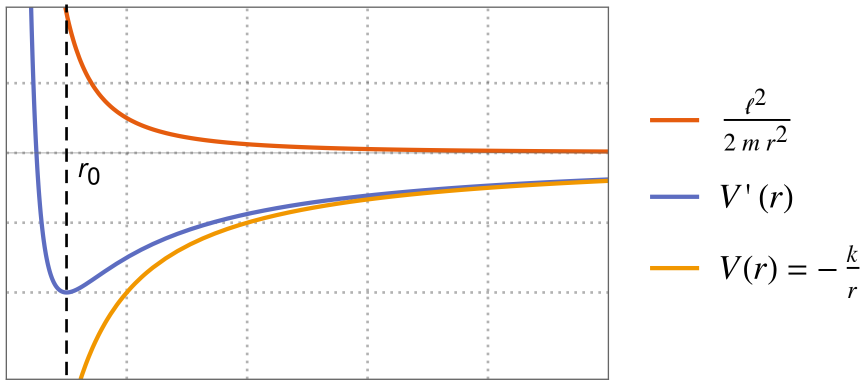


Figure 8.1: Example potential that leads to stable circular orbits.

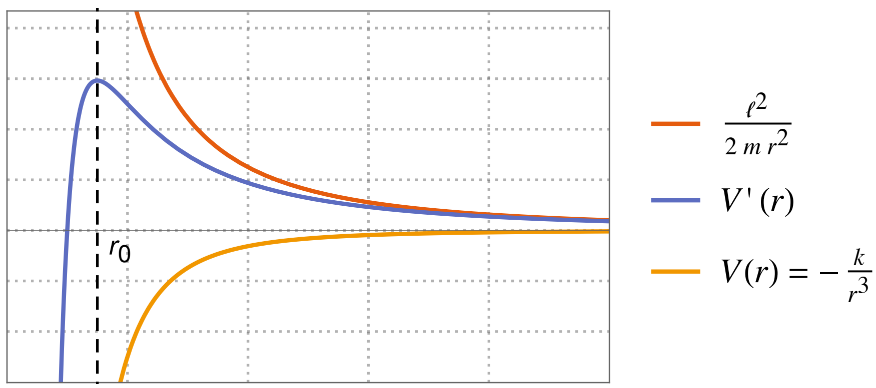


Figure 8.2: Example potential that leads to unstable circular orbits.

maximum (Fig. 8.2). The condition for the stability of a circular orbit is therefore

$$\left.\frac{d^2V'}{dr^2}\right|_{r=r_0} > 0 \quad \Rightarrow \quad -\left.\frac{dF}{dr}\right|_{r=r_0} + \frac{3\ell^2}{mr_0^4} > 0$$

Using Eq. (8.7) and keeping in mind that $F(r) < 0$ we get the condition

$$\left.\frac{dF}{dr}\right|_{r=r_0} < -3\frac{F(r_0)}{r_0} \quad \Rightarrow \quad \frac{r_0}{F(r_0)}\left.\frac{dF}{dr}\right|_{r=r_0} > -3 \tag{8.8}$$

If the force follows a power-law $F(r) = -kr^n$ then this condition reduces to $n > -3$.

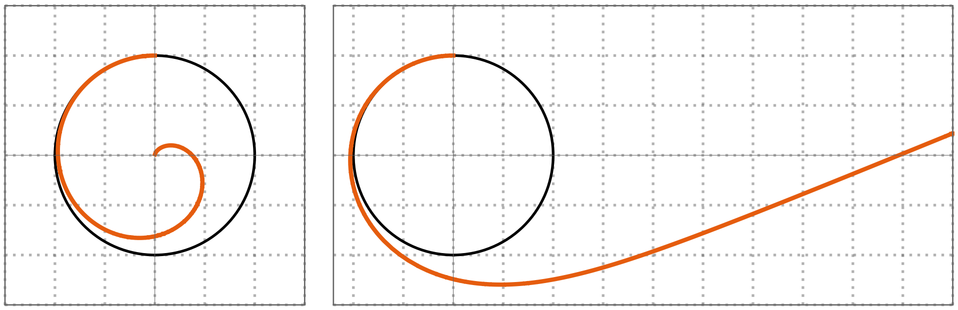


Figure 8.3: Examples of unstable circular orbits. If $V(r) = -k/r^3$, a small perturbation to a circular orbit will increase further. The body will either escape or collide with the centre.

8.1.2 Bertrand's theorem

An orbit is called **closed** if it returns onto itself, or equivalently: if $r(\vartheta)$ is a periodic function with a period that is an integer multiple of 2π. What sort of potentials will lead to a closed orbit?

It is clear that in order for the orbit to be closed, it must be bounded, and it must not collide with the centre. This happens if the effective potential $V'(r)$ has a minimum and condition (8.8) is satisfied.

Let us now derive conditions for a closed orbit for the case when the orbit differs only slightly from a circle. Defining $J(u) \equiv -\frac{m}{\ell^2}\frac{d}{du}V(\frac{1}{u})$, we can write Eq. (8.3) as

$$\frac{d^2u}{d\vartheta^2} + u = J(u).$$

Expanding $J(u)$ into Taylor series around the inverse radius of a circular orbit $u_0 = 1/r_0$, we obtain

$$\frac{d^2u}{d\vartheta^2} + u = J(u_0) + \left.\frac{dJ}{du}\right|_{u=u_0} \Delta u + \left.\frac{d^2J}{du^2}\right|_{u=u_0} \frac{\Delta u^2}{2} + \dots,$$

where $\Delta u(\vartheta) = u(\vartheta) - u_0$. Ignoring terms of second or higher order we get

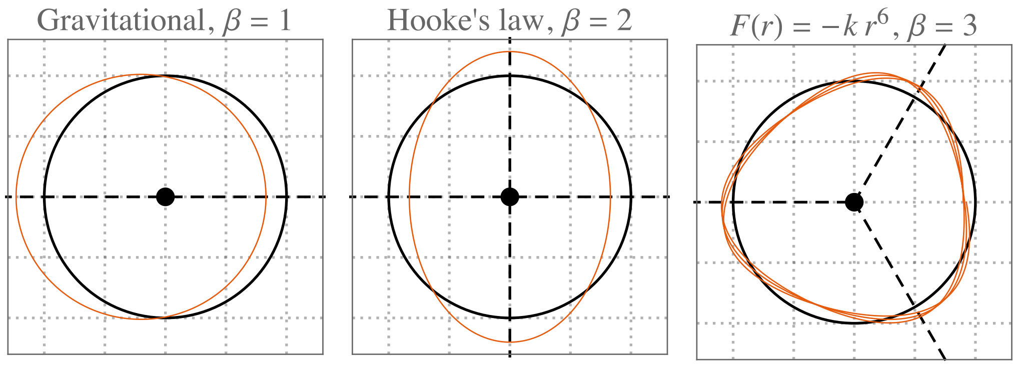


Figure 8.4: Examples of small deviations from the circular orbit for $\beta = 1, 2, 3$. $\beta = 1$ and 2 correspond to the gravitational force and Hooke's law, respectively, so the orbits are exactly closed. For $\beta = 3$ the orbit is only closed in the limit of small deviations from the circle. In this example a large enough perturbation was used to make it clear that the orbit is not exactly closed. The orbits shown are *exact* solutions of Eq. (8.3).

the approximation

$$\frac{d^2\Delta u}{d\vartheta^2} + \Delta u \underbrace{\left(1 - \left.\frac{dJ}{du}\right|_{u=u_0}\right)}_{\beta^2} \approx 0 \tag{8.9}$$

This is the familiar equation of a harmonic oscillator. In order for the orbit to be stable, β^2 must be positive, otherwise $\Delta u(\vartheta)$ would grow in magnitude exponentially. Plugging in the definition of $J(u)$ we get the following expression for β:

$$\beta^2 = 3 - \frac{u_0}{F(u_0)} \left.\frac{dF}{du}\right|_{u=u_0} = 3 + \frac{r_0}{F(r_0)} \left.\frac{dF}{dr}\right|_{r=r_0} \tag{8.10}$$

The condition that $\beta^2 > 0$ is in accordance with condition (8.8).

The general solution of equation (8.9) is

$$\Delta u(\vartheta) = \mathscr{C}\cos(\vartheta - \vartheta_0) \quad \Leftrightarrow \quad u(\vartheta) = u_0 + \mathscr{C}\cos(\vartheta - \vartheta_0), \tag{8.11}$$

where $\mathscr{C}$ is some constant. The inverse distance from the origin, u, will oscillate around u_0 with frequency β. Example orbits are shown in Fig. 8.4.

Note: Figure 3.13 in the Goldstein et al. book is incorrect. If the force is pointing towards the origin, the orbit must always curve towards the origin. See Fig. 8.4 for actual orbits computed numerically for $\beta = 1$, 2 and 3.

The orbit is closed if $r(\vartheta)$ is a periodic function with period $b \times 2\pi$, where $b \in \mathbb{Z}$ is an integer. Eq. (8.11) will have such a period if β is a rational number $\frac{a}{b} \in \mathbb{Q}$. For a given potential, β can only be a rational number if **it is a constant independent of r_0 or ℓ.** If β were a non-constant function of r_0, it would vary *continuously* based on equation 8.10, which contradicts the requirement that β is rational.

Since β is constant, we can treat Eq. (8.10) as a differential equation to find what forms of potential (or force) lead to closed orbits. The solution is a power-law:

$$F(r) = -\frac{k}{r^{3-\beta^2}} \tag{8.12}$$

We found that the orbit will be closed for any power-law potential with an exponent $n = -(3-\beta^2) > -3$ if $\beta = \sqrt{3+n}$ is a rational number. For $\beta = 1$ we obtain $n = -2$, the gravitational force. For $\beta = 2$ we obtain $n = 1$, Hooke's law. It turns out that for other values of β the orbit will only be approximately closed, for small deviations from a perfect circle.

Generally, the orbit is closed only for $\beta = 1$, $F(r) = -\frac{k}{r^2}$ and for $\beta = 2$, $F(r) = -kr$. This result was derived by the French mathematician Joseph Betrand and is known as Betrand's theorem. It can be obtained by taking into account higher order terms in the Taylor expansion of $J(u)$ and looking for the solution of Eq. (8.3) in the form of a Fourier series.

8.2 The Kepler problem

Let us compute the orbit of a body in a gravitational potential, $V(r) = -k/r$ and $F(r) = -k/r^2$. This is known as the Kepler problem. Substituting this potential into Eq. (8.5) we get the integral

$$\vartheta = \vartheta' - \int \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2mku}{\ell^2} - u^2}}. \tag{8.13}$$

Recall that $\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$. With the help of a variable change, we can compute integrals of the general form

$$\int \frac{dx}{\sqrt{\alpha+\beta x+\gamma x^2}} = \frac{1}{\sqrt{-\gamma}}\arccos\left(-\frac{\beta+2\gamma x}{\sqrt{\beta^2-4\alpha\gamma}}\right). \tag{8.14}$$

Matching up the coefficients α, β and γ with Eq. (8.24),

$$\alpha = \frac{2mE}{\ell^2}, \quad \beta = \frac{2mk}{\ell^2}, \quad \gamma = -1,$$

we get the solution

$$\vartheta = \vartheta_0 - \arccos\frac{\frac{\ell^2 u}{mk}-1}{\sqrt{1+\frac{2E\ell^2}{mk^2}}}. \tag{8.15}$$

Inverting the relationship,

$$u = \frac{1}{r} = \frac{mk}{\ell^2}\left(1+\sqrt{1+\frac{2E\ell^2}{mk^2}}\cos(\vartheta-\vartheta_0)\right) \tag{8.16}$$

This is the general equation of a conic section with eccentricity e in polar coordinates when the origin is chosen to be in the focus:

$$r = \frac{\text{const.}}{1+e\cos(\vartheta-\vartheta_0)}. \tag{8.17}$$

The eccentricity can be read off from Eq. (8.16):

$$e = \sqrt{1+\frac{2E\ell^2}{mk^2}}$$

Based on the value of the eccentricity, a conic section can be one of:

$e>1$,	$E>0$,	hyperbola
$e=1$,	$E=0$,	parabola
$e<1$,	$E<0$,	ellipse
$e=0$,	$E=-\frac{mk^2}{2\ell^2}$,	circle

In Lecture 7, Eq. (7.38), we have already computed the radius of a circular orbit, corresponding to $e=0$:

$$r_0 = \frac{\ell^2}{mk} \tag{8.18}$$

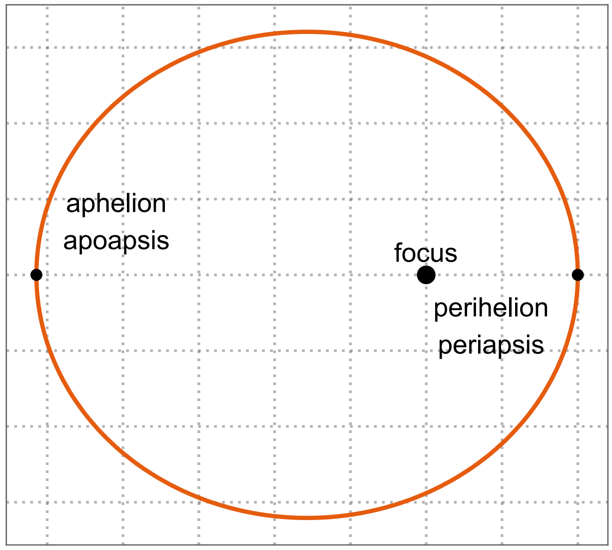


Figure 8.5: Elliptical orbit.

8.2.0.1 Elliptical orbits

If $0 \le e < 1$, then the orbit will be an ellipse. The polar equation of an ellipse in terms of its eccentricity e and half-major axis a is

$$r = \frac{a(1-e^2)}{1+\cos(\vartheta\vartheta_0)} \tag{8.19}$$

For convenience, let us choose $\vartheta_0 = 0$. Then the minimal and maximal distances of the orbit from the origin, called *apsidal distances*, will occur at $\vartheta = 0$ and $\vartheta = \pi$, respectively. These points of the orbit are called *periapsis* and *apoapsis* in general (Fig. 8.5).

> *Apoapsis* and *periapsis* are from Latin *apsis*, "arch", and Greek *apo*, "away from" and *peri*, "near to".
>
> When talking about the orbits of planets in the solar system, the common terms are *perihelion* and *aphelion*, from Greek *Helios*, "Sun". When talking about satellites of Earth, sometimes *perigee* and *apogee* are used from greek *Ge*, "Earth".

In terms of the semi-major axis, apsidal distances are $r_{1,2} = a(1 \pm e)$. We have already calculated these in terms of E and ℓ in Lecture 7, Eq. (7.37):

$$r_{1,2} = -\frac{k}{2E} \pm \frac{1}{2}\sqrt{\frac{k^2}{E^2} + \frac{2\ell^2}{mE}}$$

The semi-major axis can then be obtained as the arithmetic mean of the apsidal distances:

$$a = \frac{r_1 + r_2}{2} = -\frac{k}{2E} \tag{8.20}$$

A notable feature of this result is that the semi-major axis only depends on the total energy E, but not the angular momentum ℓ. This result has significance when discussing the Bohr model of the atom.

The eccentricity can be expressed in terms of the angular momentum ℓ and the semi-major axis as

$$e = \sqrt{1 - \frac{\ell^2}{mka}} \tag{8.21}$$

8.3 Motion in the Kepler problem

Now we have calculated the shape of the orbit, $r(\vartheta)$, in a gravitational field. Next we will compute how the body moves along this orbit in time and obtain $\vartheta(t)$, which can be substituted back into the expression of $r(\vartheta)$ to obtain $r(t)$.

$\vartheta(t)$ can be obtained as the solution of the differential equation describing the conservation of angular momentum,

$$mr(\vartheta)^2 \frac{d\vartheta}{dt} = \ell \quad \Rightarrow \quad \int_{\vartheta_0}^{\vartheta} r(\vartheta)^2 \, d\vartheta = \frac{\ell}{m} t.$$

Substituting in the expression of $r(\vartheta)$ we get the integral

$$t = \frac{\ell^3}{mk^2} \int_{\vartheta_0}^{\vartheta} \frac{d\vartheta}{(1 + e\cos\vartheta)^2} \tag{8.22}$$

Computing this integral and inverting the resulting $t(\vartheta)$ relationship is possible, but lengthy and tedious. The parabolic case is discussed in Goldstein sec. 3.8.

Let us consider the elliptic case, i.e. $0 \le e < 1$ and introduce the variable ψ defined through

$$r(\psi) = (\text{const.})(1 - e\cos\psi) \tag{8.23}$$

ψ is called the *eccentric anomaly* and was introduced originally by Kepler as an aid in his orbital calculations. The name used for our ϑ variable at the time was *true anomaly* or *equated anomaly*. Since formal equations were not commonly used in Kepler's time, he described these quantities using geometric constructions.

Over the interval $[0, 2\pi]$, ψ has a one-to-one relationship to ϑ. By considering the cases when r reaches its minimal and maximal values, as well as $r = a$, we see that

$$\begin{aligned} \psi = 0 &\Leftrightarrow \vartheta = 0 \\ \psi = \frac{\pi}{2} &\Leftrightarrow \vartheta = \frac{\pi}{2} \\ \psi = \pi &\Leftrightarrow \vartheta = \pi \end{aligned}$$

For other angles, $\psi \neq \vartheta$ when $e \neq 0$.

When written in terms of the eccentric anomaly ψ, the integral (8.22) takes a much simpler form:

$$t = \sqrt{\frac{ma^3}{k}} \int_0^{\psi} (1 - e\cos\psi)\, d\psi \tag{8.24}$$

This integral is now readily carried out to obtain **Kepler's equation**

$$\frac{2\pi}{\tau} t = \psi - e\sin\psi, \tag{8.25}$$

where τ is the period of the orbit obtained by putting $\psi = 2\pi$ in Eq. (8.24):

$$\tau = 2\pi a^{\frac{3}{2}} \sqrt{\frac{m}{k}} \tag{8.26}$$

The quantity $2\pi t/\tau$ is called the *mean anomaly*. It runs from 0 to 2π and it changes proportionally with the time t.

Now we have a way to compute the position of the body along the orbit based on the time:

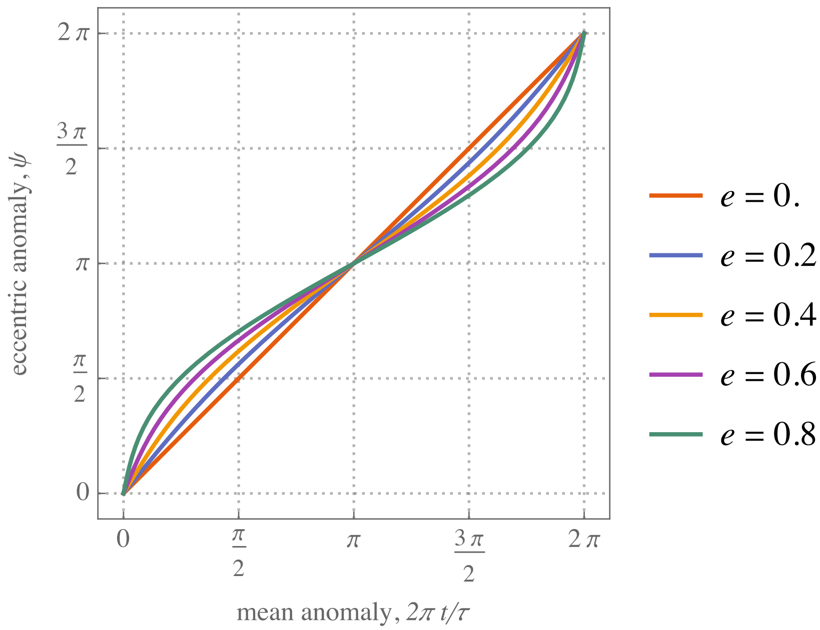


Figure 8.6: Kepler's equation relates the mean anomaly to the eccentric anomaly: $2\pi\frac{t}{\tau} = \psi - e\sin\psi$.

1. Compute the eccentric anomaly ψ from the time t (or mean anomaly) based on Kepler's equation
2. Compute the true anomaly ϑ from the eccentric anomaly.
3. Finally compute $r(\vartheta)$ based on Eq. (8.16).

Kepler's equation is a transcendental equation that is usually solved numerically. After Kepler described the problem, many mathematicians developed numerical schemes for solving this equation, including Kepler himself. The relationship between the mean and eccentric anomalies is shown in Fig. 8.6 for various values of the eccentricity e.

Equations for the relationship between ϑ and ψ can be derived based on the definition of ψ, Eq. (8.23):

$$1 - e\cos\psi = \frac{1-e^2}{1-e\cos\vartheta} \;\Leftrightarrow\; \tan\frac{\vartheta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2} \tag{8.27}$$

See Fig. 8.7 for data published by Kepler, based on Tycho Brahe's observations, connecting the eccentric anomaly and the mean anomaly.

Kepler's 3rd law

Equation 8.26 relates the semi-major axis to the period of the orbit and is an expression of **Kepler's 3rd law**. It can also be derived based on the constant areas velocity, Lecture 7, Eq. (7.17). The ray connecting the body to the centre sweeps equal areas A in unit time,

$$\frac{dA}{dt} = \frac{\ell}{2m}. \tag{8.28}$$

Integrating this relation for a full period we get

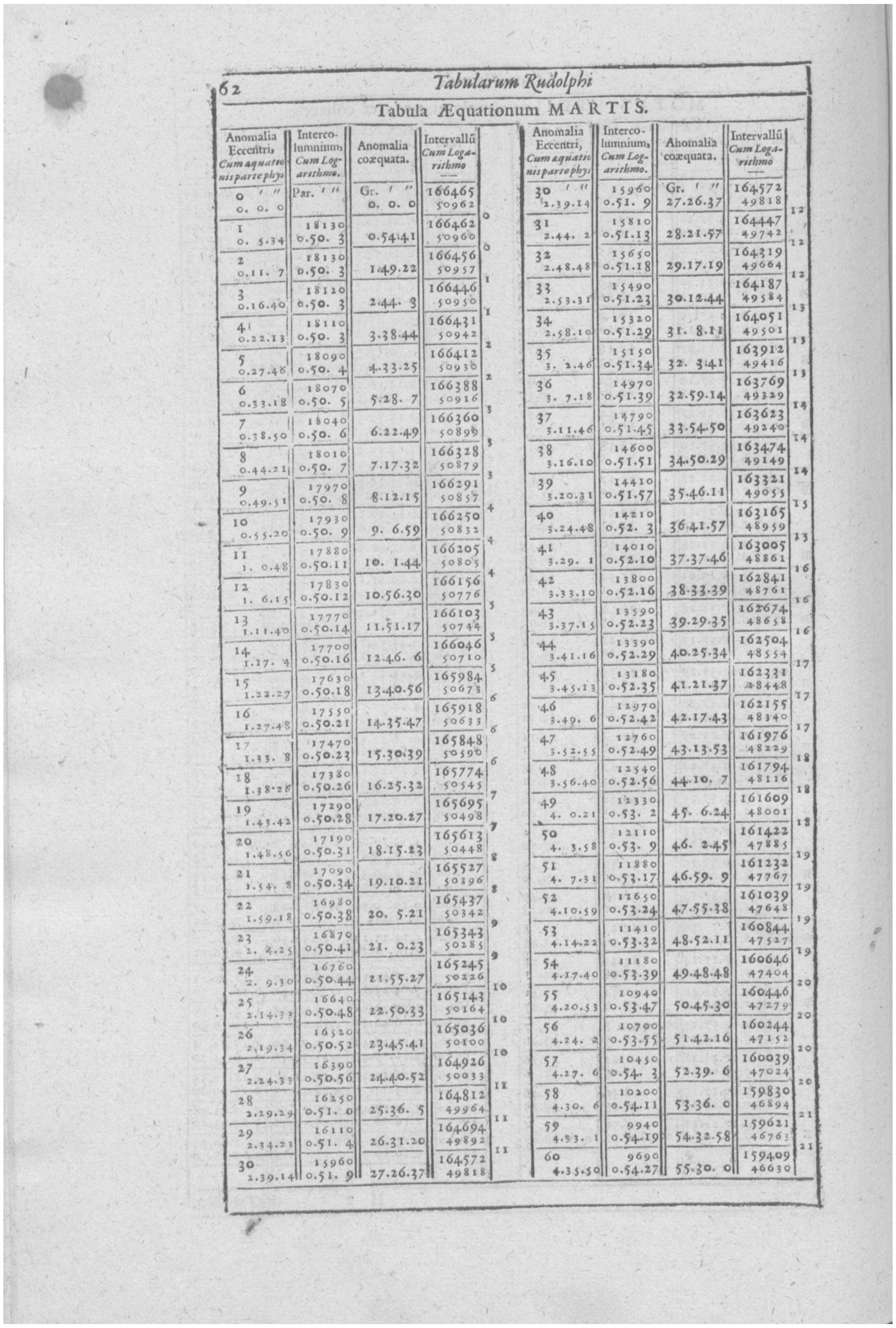
$$A = \frac{\tau \ell}{2m},$$

where A is the area of the ellipse. A can be written in terms of the semi-major and semi-minor axes, a and b, as $A = ab\pi$. The semi-minor axis can be expressed in terms of the semi-major axis and the eccentricity, $b = a\sqrt{1-e^2}$, then based on Eq. (8.21), $b = \sqrt{a}\frac{\ell}{\sqrt{mk}}$. Substituting this back into (8.28) we obtain again

$$\tau = 2\pi a^{\frac{3}{2}} \sqrt{\frac{m}{k}}$$

Kepler originally formulated his law for the periods of the planets in the solar system: the square of the period of a planet's orbit is proportional to the cube of its semi-major axis. This is equivalent to saying that the constant of proportionality $\sqrt{m/k}$ is the same for all planets. To see why this is true, remember that $m = (m_1 m_2)/(m_1 + m_2)$ is the reduced mass of the system. If $m_2 \gg m_1$ (as is the case with the mass of the Sun compared to the mass of the planets) then $m_1 \approx m$. The constant k in the gravitational force law is Gm_1m_2, so

$$\sqrt{\frac{m}{k}} = \sqrt{\frac{m}{Gm_1m_2}} \approx \sqrt{\frac{1}{Gm_2}}$$



62 *Tabularum Rudolphi*

Tabula Æquationum MARTIS.

Anomalia Eccentri,	Cum æquationis parte phys.	Intercolumnium, Cum Logarithmo.		Anomalia coæquata.	Intervallū Cum Logarithmo		
0 ′ ″	0. 0. 0	Par. ′ ″		Gr. ′ ″ 0. 0. 0	166465	50962	
1	0. 5.34	18130	0.50. 3	0.54.41	166462	50960	0
2	0.11. 7	18130	0.50. 3	1.49.22	166456	50957	0
3	0.16.40	18120	0.50. 3	2.44. 8	166446	50950	1
4	0.22.13	18110	0.50. 3	3.38.44	166431	50942	1
5	0.27.46	18090	0.50. 4	4.33.25	166412	50930	2
6	0.33.18	18070	0.50. 5	5.28. 7	166388	50916	2
7	0.38.50	18040	0.50. 6	6.22.49	166360	50899	3
8	0.44.21	18010	0.50. 7	7.17.32	166328	50879	3
9	0.49.51	17970	0.50. 8	8.12.15	166291	50857	3
10	0.55.20	17930	0.50. 9	9. 6.59	166250	50832	4
11	1. 0.48	17880	0.50.11	10. 1.44	166205	50805	4
12	1. 6.15	17830	0.50.12	10.56.30	166156	50776	4
13	1.11.40	17770	0.50.14	11.51.17	166103	50744	5
14	1.17. 4	17700	0.50.16	12.46. 6	166046	50710	5
15	1.22.27	17630	0.50.18	13.40.56	165984	50673	5
16	1.27.48	17550	0.50.21	14.35.47	165918	50633	6
17	1.33. 8	17470	0.50.23	15.30.39	165848	50590	6
18	1.38.26	17380	0.50.26	16.25.32	165774	50545	6
19	1.43.42	17290	0.50.28	17.20.27	165695	50498	7
20	1.48.56	17190	0.50.31	18.15.23	165613	50448	7
21	1.54. 8	17090	0.50.34	19.10.21	165527	50396	8
22	1.59.18	16980	0.50.38	20. 5.21	165437	50342	8
23	2. 4.25	16870	0.50.41	21. 0.23	165343	50285	9
24	2. 9.30	16760	0.50.44	21.55.27	165245	50226	9
25	2.14.33	16640	0.50.48	22.50.33	165143	50164	10
26	2.19.34	16520	0.50.52	23.45.41	165036	50100	10
27	2.24.33	16390	0.50.56	24.40.52	164926	50033	10
28	2.29.29	16250	0.51. 0	25.36. 5	164812	49964	11
29	2.34.23	16110	0.51. 4	26.31.20	164694	49892	11
30	2.39.14	15960	0.51. 9	27.26.37	164572	49818	11

Anomalia Eccentri,	Cum æquationis parte phys.	Intercolumnium, Cum Logarithmo.		Anomalia coæquata.	Intervallū Cum Logarithmo		
30 ′ ″	2.39.14	15960	0.51. 9	Gr. ′ ″ 27.26.37	164572	49818	
31	2.44. 2	15810	0.51.13	28.21.57	164447	49742	12
32	2.48.48	15650	0.51.18	29.17.19	164319	49664	12
33	2.53.31	15490	0.51.23	30.12.44	164187	49584	12
34	2.58.10	15320	0.51.29	31. 8.11	164051	49501	13
35	3. 2.46	15150	0.51.34	32. 3.41	163912	49416	13
36	3. 7.18	14970	0.51.39	32.59.14	163769	49329	13
37	3.11.46	14790	0.51.45	33.54.50	163623	49240	14
38	3.16.10	14600	0.51.51	34.50.29	163474	49149	14
39	3.20.31	14410	0.51.57	35.46.11	163321	49055	14
40	3.24.48	14210	0.52. 3	36.41.57	163165	48959	15
41	3.29. 1	14010	0.52.10	37.37.46	163005	48861	15
42	3.33.10	13800	0.52.16	38.33.39	162841	48761	16
43	3.37.15	13590	0.52.23	39.29.35	162674	48658	16
44	3.41.16	13390	0.52.29	40.25.34	162504	48554	16
45	3.45.13	13180	0.52.35	41.21.37	162331	48448	17
46	3.49. 6	12970	0.52.42	42.17.43	162155	48340	17
47	3.52.55	12760	0.52.49	43.13.53	161976	48229	17
48	3.56.40	12540	0.52.56	44.10. 7	161794	48116	18
49	4. 0.21	12330	0.53. 2	45. 6.24	161609	48001	18
50	4. 3.58	12110	0.53. 9	46. 2.45	161422	47885	18
51	4. 7.31	11880	0.53.17	46.59. 9	161232	47767	19
52	4.10.59	11650	0.53.24	47.55.38	161039	47648	19
53	4.14.22	11410	0.53.32	48.52.11	160844	47527	19
54	4.17.40	11180	0.53.39	49.48.48	160646	47404	19
55	4.20.53	10940	0.53.47	50.45.30	160446	47279	20
56	4.24. 2	10700	0.53.55	51.42.16	160244	47152	20
57	4.27. 6	10450	0.54. 3	52.39. 6	160039	47024	20
58	4.30. 6	10200	0.54.11	53.36. 0	159830	46894	20
59	4.33. 1	9940	0.54.19	54.32.58	159621	46763	21
60	4.35.50	9690	0.54.27	55.30. 0	159409	46630	21

Figure 8.7: Tables relating the eccentric anomaly, *anomalia eccentri*, ψ, with the true anomaly (also called equated anomaly), *anomalia coæquata*, ϑ, for the eccentricity of Mars; from Kepler's Rudolphine Tables, 1627.

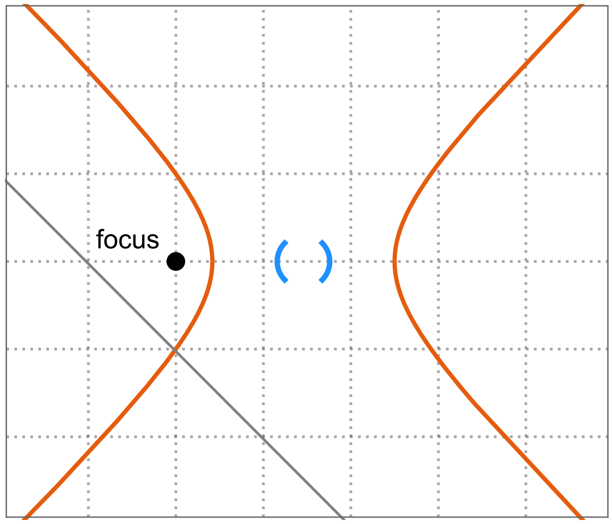


Figure 8.8: example caption

8.4 Scattering in a central force field

8.4.1 Transformation to laboratory coordinates

Reminder

Let $\mathbf{r}_1, m_1$ and $\mathbf{r}_2, m_2$ be the positions and masses of two bodies. In Lecture 7 we have replaced these coordinates by the relative position of the bodies, $\mathbf{r}$, and the position of the system's centre of mass, $\mathbf{R}$.

$$\begin{aligned} \mathbf{R} &= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2} \qquad & \mathbf{r}_1 &= \mathbf{R} + \frac{\mu}{m_1}\mathbf{r} \\ \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 & \mathbf{r}_2 &= \mathbf{R} - \frac{\mu}{m_2}\mathbf{r} \end{aligned}, \tag{8.29}$$

where

$$\mu = \frac{m_1 m_2}{m_1+m_2}$$

is the reduced mass. We have shown that if the potential energy of the system only depends on the distance of the bodies then

$$\mathbf{V} = \dot{\mathbf{R}} = \text{const.}$$

and the system is equivalent to a single body at position $\mathbf{r}$ moving in a central potential $V(|\mathbf{r}|)$.

Let us denote the coordinates in the centre of mass reference frame by $\mathbf{r}'_1$ and $\mathbf{r}'_2$. Then

$$\mathbf{v}_1 = \dot{\mathbf{r}}_1 = \dot{\mathbf{r}}'_1 + \dot{\mathbf{R}} = \mathbf{v}_1 + \mathbf{V}$$
$$\mathbf{v}_2 = \dot{\mathbf{r}}_2 = \dot{\mathbf{r}}'_2 + \dot{\mathbf{R}} = \mathbf{v}_2 + \mathbf{V}$$