Quiz Solution

PHYS205 Electricity and Magnetism

We shall assume that the magnetic field $\mathbf{B}$ and the angular velocity vector $\boldsymbol{\omega}$ point in the same direction ($\boldsymbol{\omega}\cdot\mathbf{B} > 0$). The Lorentz force causes the positive charges to move outward in the sphere, so a negative space charge will build up in the interior, while the surface charge will be positive.

The charges distribute themselves in such a way that the electric field balances the Lorentz force. The sphere is not a perfect conductor, so in a steady state the charges must move with exactly the same velocity $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ as the sphere at every point. (To keep the charges in motion relative to the sphere an electromotive force would be necessary.)

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B} \tag{1}$$

The space charge density ρ can be found by taking the divergence of this equation.

$$\begin{aligned} \rho &= \varepsilon_0 \nabla\cdot\mathbf{E} \\ &= \varepsilon_0 \nabla\cdot(\mathbf{v}\times\mathbf{B}) \\ &= \varepsilon_0(\mathbf{v}\cdot\nabla\times\mathbf{B} - \mathbf{B}\cdot\nabla\times\mathbf{v}) \end{aligned} \tag{2}$$

Here we used the fact that a mixed product does not change when its terms are cyclically permuted: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$. According to Maxwell's equations $\nabla\times\mathbf{B} = \mu_0\mathbf{j} = \mu_0\rho\mathbf{v}$.

$$\nabla\times\mathbf{v} = \nabla\times(\boldsymbol{\omega}\times\mathbf{r}) = \boldsymbol{\omega}\nabla\cdot\mathbf{r} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{r} = 2\boldsymbol{\omega}$$

Substituting these results back into equation (2), $\rho = \varepsilon_0\mu_0 v^2\rho - 2\varepsilon_0\boldsymbol{\omega}\cdot\mathbf{B}$. Using the formula $\varepsilon_0\mu_0 = 1/c^2$ we get

$$\rho(1 - v^2/c^2) = -2\varepsilon_0\boldsymbol{\omega}\cdot\mathbf{B}. \tag{3}$$

Since $v \ll c$, $1 - v^2/c^2 \approx 1$. To use equation (3) to get the space charge density, the magnetic field $\mathbf{B}$ needs to be known. But the homogeneous magnetic field will be distorted by the field originating from the circulating charges in the sphere. If the sphere is not spinning very fast, this correction is relatively small and the following method can be used to approximate $\mathbf{B}$:

First we assume that the applied homogeneous magnetic field is unchanged and denote it by $\mathbf{B}_0$, then calculate the space charge ρ and current densities $\mathbf{j} = \rho\mathbf{v}$. Next, we use these currents to find the correction $\mathbf{B}_1$ to the homogeneous field, and find the corrected charge densities and currents using $\mathbf{B}_0 + \mathbf{B}_1$. With these new current densities the next correction $\mathbf{B}_2$ can be calculated. The result can be made more accurate by doing more iterations.

Without carrying out the actual calculations, we know that $B_1 \sim \mu_0 I/R$, where I is a current-like quantity: $I \sim \omega R \rho R^2$. But $\rho \sim \varepsilon_0 \omega B_0$, so $B_1 \sim \mu_0 \varepsilon_0 \omega^2 R^2 B_0 = v^2/c^2 B_0$. Thus, when using the $v \ll c$ approximation, the corrections to the homogeneous field due to the induced rotating space charge can be neglected. Further on, $\mathbf{B}$ shall be considered homogeneous.

Now we proceed with calculating the electric field and potential *inside* the sphere. Substituting $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ into equation (1) we get $\mathbf{E} = -\mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{B}) + \boldsymbol{\omega}(\mathbf{B} \cdot \mathbf{r})$. $\boldsymbol{\omega}$ and $\mathbf{B}$ are parallel, so $\mathbf{E} = -\mathbf{a}(\boldsymbol{\omega} \cdot \mathbf{B})$, where a is the distance from the axis of rotation: if we choose $\hat{\mathbf{z}} \| \boldsymbol{\omega}$, then $\mathbf{a} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$.

It must be noted here that the electric field can compensate the Lorentz force only if $\nabla \times (\mathbf{v} \times \mathbf{B}) = 0$. This is true for a homogeneous $\mathbf{B}$-field approximation when the axis of rotation is parallel to the magnetic field, but not otherwise. If the Lorentz force can not be completely cancelled, there will always be some charge motion relative to the sphere until the energy dissipation causes the spinning sphere to stop.

The electric potential inside the sphere is

$$\phi = \frac{a^2}{2}(\boldsymbol{\omega} \cdot \mathbf{B}) + \phi_0$$

In spherical coordinates $\phi = \phi_0 - r^2 \sin^2\theta\, (\boldsymbol{\omega} \cdot \mathbf{B})/2$, where θ is the angle between $\mathbf{r}$ and the axis of rotation. Unlike $\mathbf{E}$, the potential is continuous over the surface of the sphere, so the boundary condition $\phi = \phi_0 - R^2 \sin^2\theta\, (\boldsymbol{\omega} \cdot \mathbf{B})/2$ may be used to also find ϕ *outside* the sphere of radius R.

Let us approximate a potential that satisfies this boundary condition with multipole terms! The monopole term must be zero because the total charge of the sphere is zero. The dipole term is also zero because the dipole field does not match the symmetry of this system. Therefore the dominant part of the sought potential must be a quadrupole potential. It is easy to see that the potential $\phi^{(2)}$ of a quadrupole that was created by superposing two dipoles along their axes (to satisfy the requirement of azimuthal symmetry) satisfies the boundary conditions *exactly*:

$$\phi^{(2)} = (\text{some constant})\frac{3\sin^2\theta - 2}{r^3}.$$

We need to seek no further: outside the sphere the potential will be

$$\phi = \frac{R^5(\boldsymbol{\omega}\cdot\mathbf{B})}{2r^3}\left(\sin^2\theta - \frac{2}{3}\right).$$

The surface charge density is $\sigma = \varepsilon_0(\partial_r\phi_{\text{outside}} - \partial_r\phi_{\text{inside}})\big|_{r=R}$.

$$\sigma = \varepsilon_0 R(\boldsymbol{\omega}\cdot\mathbf{B})\left(\frac{5}{2}\sin^2\theta - 1\right)$$

By integrating this over the surface of the sphere we see that indeed the total surface charge equals the total volume charge in magnitude, so the sphere has no net charge.

Summary

The potential inside the sphere is

$$\phi = \frac{a^2}{2}(\boldsymbol{\omega}\cdot\mathbf{B}) + \phi_0 = \frac{r^2\sin^2\theta}{2}(\boldsymbol{\omega}\cdot\mathbf{B}) + \phi_0.$$

The electric field inside the sphere is

$$\begin{aligned}\mathbf{E} &= -\mathbf{a}(\boldsymbol{\omega}\cdot\mathbf{B})\\ E_r &= -r\sin^2\theta\,(\boldsymbol{\omega}\cdot\mathbf{B})\\ E_\theta &= -r\sin\theta\cos\theta\,(\boldsymbol{\omega}\cdot\mathbf{B}).\end{aligned}$$

The potential outside the sphere is

$$\phi = \frac{R^5(\boldsymbol{\omega}\cdot\mathbf{B})}{2r^3}\left(\sin^2\theta - \frac{2}{3}\right).$$

The electric field outside the sphere is

$$E_r = -\frac{3R^5(\boldsymbol{\omega}\cdot\mathbf{B})}{2r^4}\left(\sin^2\theta - \frac{2}{3}\right) = \frac{R^5(\boldsymbol{\omega}\cdot\mathbf{B})}{4r^4}(1 + 3\cos 2\theta)$$

$$E_\theta = \frac{R^5(\boldsymbol{\omega}\cdot\mathbf{B})}{r^4}\cos\theta\sin\theta.$$

The surface charge density is

$$\sigma = \varepsilon_0 R(\boldsymbol{\omega}\cdot\mathbf{B})\left(\frac{5}{2}\sin^2\theta - 1\right).$$