

8th of October, 2008

1. See the book.
2. If currents are localized into a domain Ω of space, then the magnetic vector potential can be written as

$$\mathbf{A}(\mathbf{R}) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\mathbf{j}(\mathbf{r})}{|\mathbf{R} - \mathbf{r}|} d^3\mathbf{r}$$

(See fig. 1 for notations.) The first term of the multipole expansion is

$$\mathbf{A}^{(0)}(\mathbf{R}) = \frac{\mu_0}{4\pi} \frac{1}{\mathbf{R}} \int_{\Omega} \mathbf{j}(\mathbf{r}) d^3\mathbf{r}$$

Therefore it is sufficient to prove that the volume integral of the current density over Ω is 0. For this, let us calculate the divergence of $\mathbf{j}x$, where x is a position coordinate!

$$\nabla \cdot (\mathbf{j}x) = \underbrace{\mathbf{j} \cdot \nabla x}_{j_x} + \underbrace{(\nabla \cdot \mathbf{j})x}_0 = j_x$$

We know that for any current and charge distribution $\nabla \cdot \mathbf{j} + \frac{\partial}{\partial t} \rho = 0$, therefore in the stationary case $\nabla \cdot \mathbf{j} = -\frac{\partial}{\partial t} \rho = 0$. So the x component of the volume integral of \mathbf{j} can be written as

$$\int_{\Omega} j_x(\mathbf{r}) d^3\mathbf{r} = \int_{\Omega} \nabla \cdot (x\mathbf{j}(\mathbf{r})) d^3\mathbf{r} = \int_{\partial\Omega} (\mathbf{j}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}))x d^2\mathbf{r} = 0$$

Using the Gauß–Ostrogradski theorem, the volume integral on Ω can be converted into a surface integral on over the boundary $\partial\Omega$. But there are no currents outside the domain Ω , so the volume integral can be

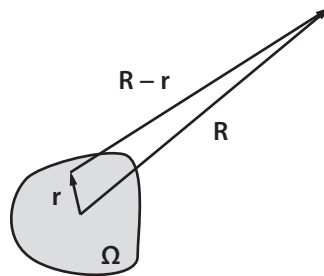
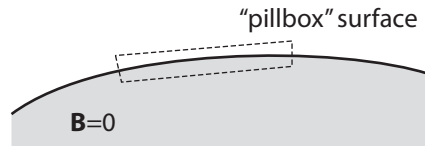


Figure 1:

extended into a slightly bigger region (one the boundary of which $\mathbf{j} = 0$), therefore the surface integral is 0.

The same reasoning can be applied to prove that the projection of the volume integral along any *any* axis is 0.

3. It can easily be shown that if there is no magnetic field inside a superconductor, then just above the surface of the superconductor the normal component of \mathbf{B} is 0. Suppose that the normal component weren't zero: in this case the surface integral of \mathbf{B} over a "pillbox" surface around the surface of the superconductor would be non-zero, which contradicts the Maxwell-equation $\nabla \cdot \mathbf{B} = 0$.



Now let us calculate the force acting on a small magnet above a superconducting plane! We can use the method of images: The normal component of the magnetic field, B_{\perp} , must be zero just above the surface. This could be achieved by removing the superconducting plane and inserting another small magnet, one that is exactly the mirror image of the original magnet with respect to the plane (see fig. 2). The magnetic field is determined at all points in space if the boundary conditions are specified. Therefore the magnetic field in the "upper" half-space will be the same in the presence of the superconductor as without the superconductor, but with the mirror magnet. The force on the magnet will be just the same as the force exerted by the mirror-magnet.

The vector potential of a magnetic monopole \mathbf{m} is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}.$$

\mathbf{r} is the position vector relative to the magnet. By taking the curl we find the magnetic field:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{r^3}$$

The energy of a magnetic dipole \mathbf{m} in a field \mathbf{B} is $W = -\mathbf{m} \cdot \mathbf{B}$, and the force acting on it is $F = -\nabla W$.

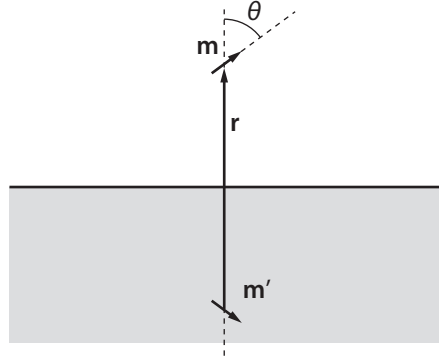


Figure 2: The magnetic dipole and its mirror image.

Let the dipole moment of the magnet be \mathbf{m} , the dipole moment of the mirror magnet be \mathbf{m}' . The energy associated with them is

$$\begin{aligned} W &= -\frac{\mu_0}{4\pi} \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})(\mathbf{m}' \cdot \hat{\mathbf{r}}) - \mathbf{m} \cdot \mathbf{m}'}{r^3} \\ &= -\frac{\mu_0}{4\pi} \frac{3m^2 \cos^2 \vartheta - m^2 \cos(2\pi - 2\vartheta)}{r^3} \\ &= -\frac{\mu_0}{4\pi} \frac{m^2}{2r^3} (3 + \cos 2\vartheta) \end{aligned}$$

The force acting on the magnet is

$$\mathbf{F} = -\nabla W = \frac{\mu_0}{4\pi} \frac{3m^2}{2r^4} (3 + \cos 2\vartheta)$$

The magnet will adopt an orientation that minimizes its energy, i.e. maximizes $\cos 2\vartheta$: $\vartheta = 0$ or $\vartheta = \pi$. The magnet's axis will be perpendicular to the superconducting surface.

4. One possible solution is to use the method of images. The image charge distribution can be constructed in the following way: take two parallel charged wires with linear charge densities $+\lambda$, and $-\lambda$. Let the distance s between the wires tend to 0 while keeping the product $\lambda s = p_\ell$ constant. It can be shown that when the field of this charge distribution (let us call it a linear dipole) is superposed onto a homogeneous field, a cylindrical equipotential surface is created (show that the resulting electric field is perpendicular to the surface of a cylinder). The value of p_ℓ must be adjusted so that we get a cylindrical equipotential of radius R .

It is a legitimate question to ask: how do we come to the conclusion that the mirror charge distribution must be a linear dipole? We saw on the lecture that a dielectric of any shape can be imagined as two superposed homogeneous charge distributions, one positive and the other negative. When there is no external field, they cancel out, but when a field is applied, they separate by a very small distance.

If we are concerned only about electrostatics, a dielectric of $\epsilon_r \rightarrow \infty$ can be considered equivalent to a conductor. This may suggest that taking two oppositely charged solid cylinders, slightly separated in a direction perpendicular to their axis, will give the correct mirror charge distribution. According to Gauß' law, outside the cylinder the field of this configuration is the same as the field of a linear dipole.

The final result is:

$$\text{potential: } V = \mathbf{E}_0 \cdot \mathbf{r}(R^2/r^2 - 1),$$

$$\text{surface charge: } \sigma = E_0 \epsilon_0 \cos \vartheta,$$

where ϑ is the angle measured from the direction of \mathbf{E}_0 along the cylinder's perimeter.