*5th of November, 2008*

1.

2. (a) A point charge is in equilibrium if the force acting on it is 0, i.e. the electric field is zero at the position of the charge.

   The equilibrium is stable if when the charge is displaced from the equilibrium position by a small distance, the forces acting on it push it back towards the equilibrium point (and not away from it).

   Suppose that there is a stable equilibrium point in a static electric field. The criterion of stability requires that the electric field point inwards on every point of a surface enclosing the equilibrium point. This would mean that the surface integral of the electric field in non-zero on this surface—a result which contradicts that the equilibrium point is in vacuum, i.e. there are no charges inside the surface. Therefore, the original assumption, that there exists a stable equilibrium point, must be false.

   (b) We can use a method similar to that employed in the previous point. In this case, the force acting on a magnetic dipole is the negative gradient of its potential energy, i.e. $\mathbf{F} = \nabla(\boldsymbol{\mu} \cdot \mathbf{B})$, where $\boldsymbol{\mu}$ is the dipole moment of the small magnet.

      i. First, let us take the case when $\boldsymbol{\mu}$ is constant (its orientation is fixed), and calculate the divergence of $\mathbf{F}$. Choosing the $z$ axis to be parallel with $\boldsymbol{\mu}$,

      $$\nabla \cdot \mathbf{F} = \nabla^2 \boldsymbol{\mu} \cdot \mathbf{B} = \mu \nabla^2 B_z.$$

      But it can be proven that the Laplcaian of any component of the magnetic field is 0 in vaccum. Taking the curl of both sides of the equation $\nabla \times \mathbf{B} = 0$, and using that $\nabla \cdot \mathbf{B} = 0$, we get

      $$\begin{aligned} \nabla \times (\nabla \times \mathbf{B}) &= 0 \\ \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} &= 0 \\ (\nabla^2 B_x, \nabla^2 B_y, \nabla^2 B_z) &= 0. \end{aligned}$$

      The divergence of $\mathbf{F}$ is zero, therefore, in analogy with the case of the electric field, a small magnet (with fixed orientation) cannot have a stable equilibrium point in a magnetic field.

      ii. Now let us take the case when the dipole may rotate freely. If there is some friction (e.g. the magnet is moving in a viscuous medium), then, given sufficient time, the magnet will adopt an orientation that is parallel with the **B** field. Its potential energy will be $W_\text{d} = -\boldsymbol{\mu}\cdot\mathbf{B} = -\mu B$. In a stable equilibrium point the potential energy has a minimum. To simplify the calculations, let us first consider the energy of a small piece of magnetizable material. Its magnetic moment is proportional to the magnetic field, $\boldsymbol{\mu} = k\mathbf{B}$, so its potential energy is $W_\text{m} = -kB^2$. Since $W_\text{m}$ can be obtained from $W_\text{d}$ by a monotonous tranformation, for a positive $k$, wherever $W_\text{m}$ has a minimum, $W_\text{d}$ has a minimum too. So the results obtained for a piece of magnetizable material (with $k > 0$) are applicable to a constant magnet too.

Let us calculate the divergence of the force acting on a piece of magnetizable material:

$$\nabla\cdot\mathbf{F}_\text{m} = -\nabla^2 W_\text{d} = k\nabla^2 B^2$$

But $\nabla^2 B^2$ can be shown to be positive or zero:

$$\begin{aligned}\nabla^2 B^2 &= \nabla^2 B_x^2 + \nabla^2 B_y^2 + \nabla^2 B_z^2\\ &= \nabla(2B_x\nabla B_x) + \cdots\\ &= 2|\nabla B_x|^2 + 2B_x\nabla^2 B_x + \cdots\\ &= 2|\nabla B_x|^2 + \cdots \geq 0,\end{aligned}$$

and therefore $\nabla\cdot\mathbf{F}_\text{m} \geq 0$.

With a reasoning similar to the one used in the previous point it can be shown that a force field having $\nabla\cdot\mathbf{F} \geq 0$ only has *unstable* equilibrium points. (The integral over the Gaussian surface drawn around the equilibrium point is positive or zero, therefore there must exist at least some points on it where **F** is pointing outwards, or **F** must be zero everywhere.)

Therefore neihter a small piece of magnetic material with $k > 0$, nor a little magnet has stable equilibrium points in a static magnetic field. But for a diamagnet $k < 0$, i.e. the magnetic moment is always oriented *antiparallel* to **B**, so a diamagnet *does* have stable equilibrium points.

3. Let us imagine that the atmosphere is divided into very thin horizontal layers, within which the index of refraction is constant.

   Let $n_i$ denote the index of refraction of the $i$th layer from the ground, and $\varphi_i$ denote the altitude of the star when viewed from that layer.

Writing Snell's law of refraction for the consecutive layers,

$$n_1 \sin(\pi/2 - \varphi_1) = n_2 \sin(\pi/2 - \varphi_2)$$
$$n_2 \sin(\pi/2 - \varphi_2) = n_3 \sin(\pi/2 - \varphi_3)$$
$$\dots$$
$$n_i \sin(\pi/2 - \varphi_i) = n_{i+1} \sin(\pi/2 - \varphi_{i+1})$$
$$\dots$$

we see that $n_1 \sin(\pi/2 - \varphi_1) = n_i \sin(\pi/2 - \varphi_i)$ for any $i$. As we go very high up, the atmosphere gets thinner, so $n$ approaches 1 and $\varphi$ approaches $\varphi_\mathrm{r}$. Thus the real altitude of the star can be obtained from the equation

$$n_\mathrm{ground} \sin(\pi/2 - \varphi_\mathrm{v}) = \sin(\pi/2 - \varphi_\mathrm{r}$$

4.

5. Let the magnitude of the charge of the ions be $q$. The electrostatic energy of one ion the electric field of others is

$$\begin{aligned} W_\mathrm{ion} &= -\frac{1}{4\pi\varepsilon_0}\frac{2q^2}{a} + \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{2a} - \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{3a} + \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{4a} - \cdots \\ &= -\frac{1}{4\pi\varepsilon_0}\frac{2q^2}{a}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right). \end{aligned}$$

The Taylor expansion of $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Substituting $x = 1$ we find that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

and therefore

$$W_\mathrm{ion} = -\frac{2}{4\pi\varepsilon_0}\frac{q^2}{a}\ln 2$$

Notice that if we add the energy of all ions in the crystal, we get *double* the energy of the complete crystal. (If we have only two ions, the energy of the complete system is the energy of *one* ion in the electric field of the other. Adding the energy of both gives twice this value.) So the electrostatic energy of the crystal per ion is

$$\frac{W}{\mathrm{ion}} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\ln 2$$

6. Water is incompressible, so according to the continuity equation $\nabla \cdot \mathbf{v} = 0$. Since $\nabla \times \mathbf{v} = 0$, there exists a scalar field $\psi$ so that $\mathbf{v} = -\nabla\psi$, and $\nabla^2\psi = 0$. This is completely analogous with electrostatics.

   Now let us examine the problem from the reference frame moving together with the ball. The electrostatic analogy still holds in this reference frame. In this reference frame the ball is not moving, but the water is flowing *past* the ball. Since the water cannot flow *into* or *out of* a solid ball, on the surface of the ball it must be true that

$$v_\perp = -\partial\psi/\partial r = 0. \tag{A}$$

   ($r$ is the radial coordinate measured from the centre of the ball.) Very far from the ball the flow is constant, $\mathbf{v} = \mathbf{v}_0$ (where $-v_0$ is the velocity of the ball in the original reference frame.)

   How should the homogeneous flow field $\mathbf{v}_0$ be changed so that condition (A) will be satisfied? We need to find a field $\mathbf{v}'$ which vanishes at infinity, cancels the normal component of $\mathbf{v}_0$ on the surface of the sphere, and satisfies $\nabla \cdot \mathbf{v}' = \nabla \times \mathbf{v}' = 0$. The sum $\mathbf{v}_0 + \mathbf{v}'$ will be the solution.

   We have seen a similar problem in electrostatics: a conducting sphere placed in a homogeneous electric field. The difference is that there the *tangential* (and not *normal*) component of the electric field needed to be cancelled on the surface of the sphere. There we found that the distortion in the field caused by the sphere is a dipole field.

   It turns out that a suitable chosen dipole field can cancel not only the tangential component of a homogeneous field on the surface of a sphere, but the normal component too. A dipole potential has the form

$$\psi = \frac{\mathbf{p} \cdot \mathbf{r}}{r^3},$$

   where $\mathbf{p}$ is a constant vector, and the dipole field is

$$\mathbf{v}' = -\nabla\psi = \frac{3(\hat{\mathbf{r}} \cdot \mathbf{p})\hat{\mathbf{r}} - \mathbf{p}}{r^3}.$$

   Let us choose $\mathbf{p}$ so that it is parallel to $\mathbf{v}_0$ and calculate the normal components of $\mathbf{v}_0$ and $\mathbf{v}'$ at the surface of the sphere. We shall use spherical coordinates, with $\theta$ being the angle between $\mathbf{p}$ and $\mathbf{r}$. Let $R$ be the radius of the sphere.

$$\begin{aligned} v_{0\perp} &= \mathbf{v}_0 \cdot \hat{\mathbf{r}} = v_0 \cos\theta \\ v'_\perp &= \mathbf{v}' \cdot \hat{\mathbf{r}} = \frac{3p\cos\theta - p\cos\theta}{R^3} = \frac{2p}{R^3}\cos\theta \end{aligned}$$

If $p$ is chosen so that $2p/R^3 = -v_0$ then the $\mathbf{v}'$ will cancel the normal component of $\mathbf{v}_0$ on the surface of the sphere. The solution is

$$\mathbf{v} = \mathbf{v}_0 - \frac{R^3}{2r^3}\left(3(\hat{\mathbf{r}} \cdot \mathbf{v_0})\hat{\mathbf{r}} - \mathbf{v_0}\right)$$

in the reference frame moving together with the ball.